ON THE ISOTRIVIALITY OF FAMILIES OF ELLIPTIC SURFACES

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A family $f : X \to B$ of projective complex manifolds is called birationally isotrivial, if there exists a finite cover $B' \to B$, a manifold F and a birational map φ from $F \times B'$ to $X \times_B B'$. The morphism f is isotrivial, if φ can be chosen to be biregular.

One can ask, tempted by the corresponding property for families of curves, whether f is birationally isotrivial whenever B is an elliptic curve or $\mathbb{C}^*$ and the Kodaira dimension of a general fibre non-negative. Assuming that all fibres of f are minimal models, one could even hope that f is isotrivial.

Both problems have an affirmative answer, if local Torelli theorems hold true for the fibres of f (or, as explained in 1.4, for some étale cover), and both have been solved by Migliorini [13] and Kovács [10] for families of surfaces of general type (see also [21], [4] or [2]). In this note we want to extend their methods to surfaces of Kodaira dimension one and thereby complete the proof of the following theorem.

Theorem 0.1. *All smooth projective families of minimal surfaces of non-negative Kodaira dimension over complex elliptic curves or over $\mathbb{C}^*$ are isotrivial.*

The projectivity assumption is essential. Indeed there exist smooth, highly non-projective families of $K3$-surfaces over $\mathbb{P}^1$, called twistor spaces.

Let M_h be the quasi-projective moduli scheme of polarized manifolds with numerically effective canonical divisor and Hilbert polynomial h (see [20]). If Y is a complex algebraic manifold, $\Phi : Y \to M_h$ a morphism, étale over its image, and if Φ is induced by a "universal" family, then 0.1 implies that Y is algebraically hyperbolic for $\deg(h) = 2$ (see also [11]).

If $\bar{Y}$ is a smooth compactification with $S = \bar{Y} - Y$ a normal crossing divisor, one might hope, that $\Omega^1_{\bar{Y}}(\log S)$ (or some symmetric product) contains a subbundle $\mathcal{F}$, isomorphic to $\Omega^1_{\bar{Y}}(\log S)$ over Y, with $\mathcal{F}$ numerically effective and $\det(\mathcal{F})$ big. This positivity property holds true for moduli schemes of curves, and it has recently been verified by Zuo [22] if the fibres of the universal family over Y satisfy the local Torelli theorem.

If B is an elliptic curve, or if the fibres X_b of f allow an étale cover which is an elliptic surface without multiple fibres, the proof of the isotriviality is quite easy. In the first case, the proof is given at the beginning of section 4, in the second the necessary arguments are sketched in 4.2 and 4.3, as special cases of the proof of 0.1 for elliptic surfaces, given in section 7.

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Notations 0.2. In discrepancy to the introduction X and B will denote complex projective manifolds of dimension three and one, and $f : X \to B$ will be a family of surfaces, i.e. a flat projective morphism with two dimensional connected fibres $X_b = f^{-1}(b)$. We fix an open dense subscheme $B_0 \subset B$, such that

$$f_0 = f|_{X_0} : X_0 = f^{-1}(B_0) \longrightarrow B_0$$

is smooth, and we write $S = B - B_0$ and $\Delta = f^*(S)$.

We will call f a family of minimal surfaces, if the non-singular fibres X_b, for $b \in B_0$, are minimal models of non-negative Kodaira dimension, but we will not require f to be a relative minimal model in a neighborhood of $f^{-1}(S)$.

The dualizing sheaves of B, X and of f will be denoted by ω_B, ω_X and $\omega_{X/B} = \omega_X \otimes f^*\omega_B^{-1}$.

If D is an effective normal crossing divisor on X, $\Omega^i_X(\log D) = \Omega^i_X(\log D_{\mathrm{red}})$ denotes the sheaf of logarithmic differential forms.

Starting from section three, the general fibre F of f is assumed to be a minimal elliptic surface of Kodaira dimension $\kappa(F) = 1$ and starting with section four, we will assume that B is an elliptic curve and $S = \emptyset$, or that $(B, S) = (\mathbb{P}^1, \{0, \infty\})$.

1. Families of surfaces and isotriviality

The positivity results for direct images of powers of dualizing sheaves, due to Fujita, Kawamata and the second named author (see [15], 7.2 and the references given there) can be presented in a nice form, if the base is a curve and if the smooth fibres are minimal.

Definition 1.1. Let X be a projective manifold and $U \subset X$ an open dense subset. An invertible sheaf $\mathcal{L}$ on X is called

i) semi-ample with respect to U, if for some μ_0 and all multiples μ of μ_0 the map
$$\varphi_\mu : H^0(X, \mathcal{L}^\mu) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \mathcal{L}^\mu$$
is surjective over U.

ii) ample with respect to U, if $\mathcal{L}$ is semi-ample with respect to U and if φ_μ induces an embedding $U \to \mathbb{P}(H^0(X, \mathcal{L}^\mu))$ for μ sufficiently large.

Lemma 1.2. *Let $f : X \to B$ be a family of minimal surfaces of non-negative Kodaira dimension, smooth over $B_0 = B - S$.*

a) *Then $f_*\omega^\nu_{X/B}$ is numerically effective, for all $\nu \geq 1$.*

b) *If f is semi-stable, then the following conditions are equivalent:*

   i) *For some $\nu_0 > 0$ and for all multiples ν of ν_0 $f_*\omega^\nu_{X/B}$ is ample.*

   ii) *There exists some $\eta > 0$ such that $f_*\omega^\eta_{X/B}$ contains an ample subsheaf.*

   iii) *$\omega_{X/B}$ is semi-ample with respect of $X_0 = X - f^{-1}(S)$ and for a general fibre F of f one has $\kappa(\omega_{X/B}) = \kappa(F) + 1$.*

   iv) *f is not birationally isotrivial.*

Corollary 1.3. *Let $\tau : Y \to X$ be generically finite. If $f \circ \tau : Y \to B$ is birationally isotrivial, then the same holds true for $f : X \to B$.*

Proof. We may assume both, f and $f \circ \tau$ to be semi-stable. The natural inclusion $\omega_{X/B} \to \tau_* \omega_{Y/B}$ induces an inclusion $f_* \omega^{\nu}_{X/B} \to (f \circ \tau)_* \omega^{\nu}_{Y/B}$, for all $\nu > 0$. Hence if f is not birationally isotrivial, the condition ii) in 1.2 b) is satisfied. □

For a smooth projective family $f_0 : X_0 \to B_0$ consider the polarized variation of Hodge-structures $R^2 f_{0*} \mathbb{C}_{X_0}$. If B_0 is an elliptic curve or $\mathbb{C}^*$, then this variation of Hodge-structures is necessarily trivialized over some étale cover $B'_0 \to B_0$. In fact, the induced morphism from the universal cover $\mathbb{C}$ of B_0 to the period domain of polarized Hodge-structures is constant (see for example [19], §3). Combined with 1.3 one obtains:

Corollary 1.4. *If there exists an étale covering $\tau_0 : Y_0 \to X_0$, such that the fibres of $f_0 \circ \tau_0$ satisfy the local Torelli theorem, and if B_0 is an elliptic curve over $\mathbb{C}^*$, then f is birationally isotrivial.*

Remark 1.5. The assumptions of 1.4 hold true for all families of minimal surfaces of Kodaira dimension zero. The same argument can be used to prove the corresponding statement for families of curves of genus $g \geq 1$.

For families of minimal surfaces the birational isotriviality is equivalent to the isotriviality. As well-known, the trivialization even exists over an étale cover of B_0.

Lemma 1.6. *A smooth projective family $f_0 : X_0 \to B_0$ of minimal surfaces (or curves) of non-negative Kodaira dimension is birationally isotrivial, if and only if there exists a finite étale cover $B'_0 \to B_0$ and a surface (or curve) F with*

$$X_0 \times_{B_0} B'_0 \simeq F \times B'_0.$$

Proof. It is easy to find a finite cover $B''_0 \to B_0$ and an isomorphism

$$\varphi : X_0 \times_{B_0} B''_0 \xrightarrow{\sim} F \times B''_0$$

of polarized manifolds. In fact, there exists a coarse moduli space M_h of polarized manifolds, and Kollár and Seshadri constructed a finite cover of M_h which carries a universal family (see [20], p. 298). Of course one may assume $B''_0 \to B_0$ to be Galois with group G. In different terms, one has a lifting of the Galois action on B''_0 to $F \times B''_0$, giving X_0 as a quotient. Let H be the ramification group of a point $b \in B''_0$. Then H acts trivially on the fibre $F \times \{b\}$.

On the other hand, the automorphism group of a polarized manifold of non-negative Kodaira dimension is finite, hence the action of H on $F \times B''_0$ must locally be the pullback under pr_2 of the action on B''_0. Necessarily the same holds true globally and

$$X_0 \times_{B_0} (B''_0/H) = (F \times B''_0)/H = F \times (B''_0/H).$$

□

2. A vanishing theorem

As in [13], [10], [21], [4] or [2] we will use vanishing theorems for the cohomology of differential forms with logarithmic poles. However, we have to allow poles along some divisor Π, transversal to the elliptic fibration. In order to find such a divisor, we will be forced to modify f_0 and to allow some additional singular points in the fibres.

Assumption 2.1. Let X, W and B be normal proper algebraic varieties of dimension three, two and one respectively, and let

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & W \arrow[dl, "h"] \\ & B & \end{tikzcd}$$

be morphisms with connected fibres. Consider an effective divisor Υ and a prime divisor Π on X, and an invertible sheaf $\mathcal{L}$ on W. Let $B_0 = B - S$ be open and dense in B,

$$X_0 = f^{-1}(B_0), \qquad W_0 = h^{-1}(B_0)$$

and denote by f_0, g_0, Π_0, $\mathcal{L}_0$ and h_0 the restrictions to X_0 and W_0, respectively. Assume:

i) Π_0 is a section, i.e. $g|_{\Pi_0} : \Pi_0 \to W_0$ is an isomorphism.
ii) X is non-singular and $\Delta = f^*(S)$ as well as $\Delta + \Pi$ are normal crossing divisors.
iii) $h_0 : W_0 \to B_0$ is smooth.
iv) $g_0 : X_0 \to W_0$ is a flat family of curves.
v) $f_0 : X_0 \to B_0$ is smooth outside of a finite subset T of X_0.
vi) The sheaf $\mathcal{L}$ is ample with respect to W_0.
vii) $h_*\mathcal{L}^\nu \cong f_*(g^*\mathcal{L}^\nu \otimes \mathcal{O}_X(-\nu \cdot \Upsilon))$, for all $\nu > 0$. In particular Υ is supported in Δ.
viii) $\deg \omega_B(S) \geq 0$.

Definition 2.2.

a) For $\iota : X - T \to X$ define $\Omega^i_{X/B}(\log \Delta)^\sim = \iota_* \Omega^i_{X-T/B}(\log \Delta)$.
b) $\Omega^i_{X/B}(\log \Delta)' = \mathrm{Im}(\Omega^i_X(\log \Delta) \longrightarrow \Omega^i_{X/B}(\log \Delta)^\sim)$.
c) We use the same notation for the sheaves of differential forms with logarithmic poles along Π:

$$\Omega^i_{X/B}(\log(\Delta + \Pi))^\sim = \iota_* \Omega^i_{X-T/B}(\log(\Delta + \Pi)) \qquad \text{and}$$
$$\Omega^i_{X/B}(\log(\Delta + \Pi))' = \mathrm{Im}(\Omega^i_X(\log(\Delta + \Pi)) \to \Omega^i_{X/B}(\log(\Delta + \Pi))^\sim)$$

Since Π does not meet the non-smooth locus T of f_0, the sheaf

$$\Omega^2_{X/B}(\log(\Delta + \Pi))'$$

is invertible in a neighborhood of Π and

$$\Omega^2_{X/B}(\log(\Delta + \Pi))' = \Omega^2_{X/B}(\log \Delta)' \otimes \mathcal{O}_X(\Pi). \tag{2.2.1}$$

By definition one has the exact sequences

$$0 \longrightarrow f^*\omega_B(S) \longrightarrow \Omega^1_X(\log(\Delta + \Pi)) \longrightarrow \Omega^1_{X/B}(\log(\Delta + \Pi))' \longrightarrow 0 \tag{2.2.2}$$

$$\begin{aligned} 0 \longrightarrow f^*\omega_B(S) \otimes \Omega^1_{X/B}(\log(\Delta + \Pi))^\sim \longrightarrow \Omega^2_X(\log(\Delta + \Pi)) \longrightarrow \\ \longrightarrow \Omega^2_{X/B}(\log(\Delta + \Pi))' \longrightarrow 0. \end{aligned} \tag{2.2.3}$$

The main result of this section is

Proposition 2.3. *Assuming 2.1*

$$H^0(X,\Omega^2_{X/B}(\log(\Delta+\Pi))'\otimes g^*\mathcal{L}^{-1}\otimes\mathcal{O}_X(\Upsilon-\Pi)\otimes f^*\omega_B(S)^{-2})=$$
$$H^0(X,\Omega^2_{X/B}(\log\Delta)'\otimes g^*\mathcal{L}^{-1}\otimes\mathcal{O}_X(\Upsilon)\otimes f^*\omega_B(S)^{-2})=0.$$

Remark 2.4. If f is semistable, $\Omega^2_{X/B}(\log\Delta)^\sim=\omega_{X/B}$ and $\Omega^2_{X/B}(\log\Delta)'$ is a subsheaf, say $\omega'_{X/B}$, of $\omega_{X/B}$. Then 2.3 says that

$$H^0(X,\omega'_{X/B}(\Upsilon)\otimes f^*\omega_B(S)^{-2}\otimes g^*\mathcal{L}^{-1})=0.$$

Proof of 2.3. The statement is compatible with blowing up W and X, as long as the centers are contained in $h^{-1}(S)$ and $f^{-1}(S)$, respectively. In fact, for $\tau:X'\to X$ and $\Delta'=\tau^*\Delta$

$$\Omega^2_{X/B}(\log\Delta)'\otimes\mathcal{O}_X(\Upsilon)=\tau_*(\Omega^2_{X'/B}(\log\Delta')'\otimes\mathcal{O}_{X'}(\tau^*\Upsilon)).$$

Blowing up W (and hence X) we may assume that W is non-singular. For μ sufficiently large, $\mathcal{L}^\mu(-h^*(S)_{\mathrm{red}})$ is ample with respect to W_0.

Hence, blowing up W and replacing μ by some multiple, we will find an effective divisor Σ in W such that $\mathcal{L}^\mu(-\Sigma)$ is globally generated and big, and such that $\Sigma_{\mathrm{red}}=h^*(S)_{\mathrm{red}}$. Moreover, if $\eta:W\to\mathbb{P}(H^0(W,\mathcal{L}^\mu(-\Sigma))$ denotes the induced morphism, we can also assume that there exists an effective relatively anti-ample exceptional divisor E. Replacing $\mathcal{L}^\mu(-\Sigma)$ by $\mathcal{L}^{\mu\cdot\nu}(-\nu\cdot\Sigma-E)$, we may assume finally that $\mathcal{L}^\mu(-\Sigma)$ is ample. The assumption 2.1, vii), implies that $g^*\Sigma\geq\mu\cdot\Upsilon$.

Since Π_0 is a section, for some $\rho>0$ the map

$$g^*g_*\mathcal{O}_X(\rho\cdot\Pi)\longrightarrow\mathcal{O}_X(\rho\cdot\Pi)$$

is surjective over X_0. After blowing up X, one finds an effective divisor Γ_1, supported in Δ, with

$$g^*g_*\mathcal{O}_X(\rho\cdot\Pi)\twoheadrightarrow\mathcal{O}_X(\rho\cdot\Pi-\Gamma_1).$$

Let $\Sigma_1,\ldots,\Sigma_r$ be the irreducible components of Σ. For all ν, sufficiently large, and for all $\Sigma'=\sum_{i=1}^r\epsilon_i\Sigma_i\geq 0$, with $\epsilon_i\in\{0,1\}$,

$$g^*(\mathcal{L}^{\mu\cdot\nu})\otimes\mathcal{O}_X(\rho\cdot\Pi-g^*(\nu\Sigma+\Sigma')-\Gamma_1)$$

is big and generated by its global sections. Choosing ν larger than ρ and larger than the multiplicities of the components of $g^*(\Sigma_{\mathrm{red}})$ one finds $\epsilon_1,\ldots,\epsilon_r$ such that $N=\nu\cdot\mu$ does not divide the multiplicities of the components of

$$\Gamma=g^*(\nu\cdot\Sigma+\Sigma')+\Gamma_1.$$

By construction $\Gamma_{\mathrm{red}}=\Delta_{\mathrm{red}}$, $\Gamma\geq N\cdot\Upsilon$, and $g^*(\mathcal{L}^N)\otimes\mathcal{O}_X(\rho\cdot\Pi-\Gamma)$ is globally generated and big. Let us write

$$\mathcal{L}'=g^*(\mathcal{L})\otimes\mathcal{O}_X\Big(\Pi-\Big[\frac{\Gamma}{N}\Big]\Big)=g^*(\mathcal{L})\otimes\mathcal{O}_X\Big(\Pi-\Big[\frac{(N-\rho)\cdot\Pi+\Gamma}{N}\Big]\Big).$$

Claim 2.5. For all $m\geq 0$ and for $i+j<3$,

$$H^i(X,\Omega^j_X(\log(\Delta+\Pi))\otimes\mathcal{L}'^{-1}\otimes f^*\omega_B(S)^{-m})=0.$$

Before proving 2.5, let us deduce 2.3. Using 2.5 and the long exact cohomology sequence induced by (2.2.3) $\otimes {\mathcal L'}^{-1} \otimes f^*\omega_B(S)^{-2}$ one obtains an embedding of

$$\begin{aligned} H^0 &:= H^0(X, \Omega^2_{X/B}(\log(\Delta+\Pi))' \otimes g^*\mathcal{L}^{-1} \otimes \mathcal{O}_X\Big(-\Pi + \Big[\frac{\Gamma}{N}\Big]\Big) \otimes f^*\omega_B(S)^{-2}) \\ &= H^0(X, \Omega^2_{X/B}(\log(\Delta+\Pi))' \otimes {\mathcal L'}^{-1} \otimes f^*\omega_B(S)^{-2}) \end{aligned}$$

into

$$H^1 := H^1(X, \Omega^1_{X/B}(\log(\Delta+\Pi))^{\sim} \otimes {\mathcal L'}^{-1} \otimes f^*\omega_B(S)^{-1}).$$

Since $\Omega^1_{X/B}(\log(\Delta+\Pi))' \to \Omega^1_{X/B}(\log(\Delta+\Pi))^{\sim}$ is surjective outside of a finite set of points, H^1 is a quotient of

$$H'^1 := H^1(X, \Omega^1_{X/B}(\log(\Delta+\Pi))' \otimes {\mathcal L'}^{-1} \otimes f^*\omega_B(S)^{-1}).$$

Applying 2.5, for $j=1$, $i=1$, to (2.2.2), one finds an injective map

$$H'^1 \longrightarrow H^2(X, {\mathcal L'}^{-1})$$

and, 2.5, for $j=0$, $i=2$, implies that both groups are zero. Hence all the groups, H'^1, H^1 and H^0, are zero. Since $\left[\frac{\Gamma}{N}\right] \geq \Upsilon$ one obtains 2.3 from $H^0 = 0$. □

Proof of 2.5. By the choice of $\mathcal{L}'$ one has

$$g^*(\mathcal{L}^N) \otimes \mathcal{O}_X(\rho \cdot \Pi - \Gamma) = {\mathcal L'}^N \otimes \mathcal{O}_X(-(N-\rho)\cdot \Pi - \Gamma')$$

for $\Gamma' = \Gamma - N \cdot \left[\frac{\Gamma}{N}\right]$. Since N does not divide the multiplicities of the components of Γ, one finds $\Gamma'_{\rm red} = \Delta_{\rm red}$. The sheaf $g^*(\mathcal{L}^N) \otimes \mathcal{O}_X(\rho\cdot\Pi - \Gamma)$ contains the inverse image of an ample invertible sheaf on W. All this remains true, if we replace $\mathcal{L}'$ by $\mathcal{L}' \otimes f^*\omega_B(S)^m$, and $\mathcal{L}$ by $\mathcal{L} \otimes h^*\omega_B(S)^m$. So we may assume m to be zero.

If $\delta : X \to \mathbb{P}^M$ is the morphism given by the global sections of the ν-th power of $g^*(\mathcal{L}^N) \otimes \mathcal{O}_X(\rho \cdot \Pi - \Gamma)$, for ν sufficiently large, then $\delta|_{X - \Gamma_{\rm red}} = \delta|_{X_0}$ can at most contract components of the fibres of g_0. In particular the maximal fibre dimension of $\delta|_{X_0}$ is one.

The divisor H of a general section of $g^*(\mathcal{L}^{N\cdot\nu}) \otimes \mathcal{O}_X(\rho\cdot\Pi - \Gamma)^\nu$ is smooth, and $\Pi + \Gamma + H$ a normal crossing divisor. By [5], 6.2 a) and 4.11 b),

$$H^i(X, \Omega^j_X(\log(\Pi+\Gamma+H)) \otimes {\mathcal L'}^{-1}) = 0$$

for $i+j \neq 3$, and

$$H^i(H, \Omega^j_H(\log(\Pi+\Gamma)|_H) \otimes {\mathcal L'}^{-1}) = 0,$$

for $i+j \neq 2$. Considering the long exact sequence for

$$\begin{aligned} 0 \longrightarrow \Omega^j_X(\log(\Pi+\Gamma)) \otimes {\mathcal L'}^{-1} \longrightarrow \Omega^j_X(\log(\Pi+\Gamma+H)) \otimes {\mathcal L'}^{-1} \\ \longrightarrow \Omega^{j-1}_H(\log((\Pi+\Gamma)|_H) \otimes {\mathcal L'}^{-1} \longrightarrow 0 \end{aligned}$$

one obtains 2.5. □

3. Families of elliptic surfaces

Let us return to the family $f : X \to B$ of minimal elliptic surfaces of Kodaira dimension one, with $f_0 : X_0 \to B_0$ smooth. By [12] or [16], for all $\nu \geq 0$ and $b \in B_0$ with $X_b = f^{-1}(b)$,

$$f_* \omega^\nu_{X/B} \otimes \mathbb{C}(b) = H^0(X_b, \omega^\nu_{X_b}). \tag{3.0.1}$$

Fibrewise, for ν sufficiently large and divisible, $H^0(X_b, \omega^\nu_{X_b})$ defines the Iitaka map $X_b \to W_b$ to a non-singular curve W_b, and by (3.0.1)

$$f^* f_* \omega^\nu_{X/B} \longrightarrow \omega^\nu_{X/B}$$

defines the relative Iitaka map

$$X \xrightarrow{g} W \subset \mathbb{P}(f_* \omega^\nu_{X/B}),$$

whose restriction g_0 to X_0 is a morphism, and $W_0 = g(X_0)$ is smooth over B_0.

Blowing up X, as always with centers in $f^{-1}(S)$, we can factor f as

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & W \arrow[dl, "h"] \\ & B & \end{tikzcd}$$

where X, W and B are non-singular projective manifolds and where $g_0 : X_0 \to W_0$ is a flat projective family of curves. Using the corresponding property for $X_b \to W_b$, one finds

$$\dim H^i(g^{-1}(w), \omega^\nu_{g^{-1}(w)}) = 1,$$

for $i = 0, 1$ and $w \in W_0$. Hence $g_{0*}\omega^\nu_{X_0/W_0}$ is invertible for all $\nu \geq 0$. Moreover,

$$g_0^* g_{0*} \omega_{X_0/W_0} = \omega_{X_0/W_0}(-\tilde{\Gamma}^{(0)})$$

for some divisor $\tilde{\Gamma}^{(0)}$.

We will need several properties of elliptic threefolds, i.e. threefolds with an elliptic fibration. The results needed, due to Kawamata, Fujita, Nakayama, Miranda, Dolgachev-Gross and Gross are recalled in [8], together with more precise references. For elliptic threefolds occurring as the total space of a family of elliptic surfaces, [7] is an excellent source. The properties and definitions needed from the theory of elliptic surfaces, in particular Kodaira's classification of the singular fibres, can be found in [1].

By [8], Lemma 1.2, blowing up W with centers in $W - W_0$ one finds a flat relative minimal model $g_m : X_m \to W$, extending $g_0 : X_0 \to W_0$. Before stating this result in 3.2, we will use it to define the multiple locus and the discriminant divisor. In fact to this aim it would be sufficient to know the existence of g_m over a subscheme W_1 with $\mathrm{codim}(W - W_1) \geq 2$.

Let $\Delta(g_m)$ be the smallest subvariety such that

$$g_m^{-1}(W - \Delta(g_m)) \longrightarrow W - \Delta(g_m)$$

is smooth.

Notations 3.1. An irreducible one-dimensional component of $\Delta(g_m)$ belongs to one of the following, according to the fibre $E = g_m^{-1}(w)$ over the general point w of the component:

a) E is a multiple fibre. We denote those components by $\Sigma_1, \ldots, \Sigma_r$ and call $\Sigma = \sum_{i=1}^{r} \Sigma_i$ the multiple locus. To Σ_i we attach the multiplicity m_i of the general fibre, and $\Gamma_i = g_m^{-1}(\Sigma_i)_{\mathrm{red}}$, hence $m_i \cdot \Gamma_i = g_m^{-1}(\Sigma_i)$.
b) Let $j : W \to \mathbb{P}^1$ denote the rational map, induced by the j-invariant. Let $D_1, \ldots, D_s$ be the components of the discriminant locus whose image is ∞. To D_i we attach the multiplicity b_i of D_i in $j^{-1}(\infty)$. In particular, if the general fibre over D_i is a Newton polygon, then b_i is nothing but the length of the polygon, (i.e. type I_{b_i}). We write $J_\infty = \sum_{i=1}^{s} b_i D_i$.
c) If E is not a multiple fibre, nor a Newton polygon, we denote the corresponding components by $D_{s+1}, \ldots, D_\ell$ and we attach a number b_i to D_i according to Kodaira's classification (see [1], for example):

type	I_n^*	II	III	IV	II*	III*	IV*
b_i	6	2	3	4	10	9	8

d) $D = \bigcup_{i=1}^{\ell} D_i$ will be called the discriminant locus, and

$$\sum_{i=1}^{\ell} b_i D_i = J_\infty + \sum_{i=s+1}^{\ell} b_i D_i$$

the discriminant divisor.

Remark that Σ and J_∞ can have common components, corresponding to ${}_mI_n$. The component of the discriminant locus with general fibre of type I_n^* will occur in $J_\infty = \sum_{i=1}^{s} b_i D_i$ with multiplicity n and in $\sum_{i=s+1}^{\ell} b_i D_i$ with multiplicity 6.

Lemma 3.2. *Blowing up W with centers in $W - W_0$, there exists a flat morphism $g_m : X_m \to W$, with $g_m^{-1}(W_0) = X_0$ and $g_m|_{X_0} = g_0$, such that*
a) *$W - W_0$ is a normal crossing divisor.*
b) *X_m has at most $\mathbb{Q}$-factorial terminal singularities.*
c) *$g_{m*}\omega_{X_m/W}$ is an invertible sheaf δ.*
d) *$\delta^{12} \simeq \mathcal{O}_W(\sum_{i=1}^{\ell} b_i D_i) = \mathcal{O}_W(J_\infty + \sum_{i=s+1}^{\ell} b_i D_i)$, and, for all $\nu \geq 0$*

$$\omega_{X_m/W}^{[\nu]} = g_m^* \delta^\nu \otimes \mathcal{O}_{X_m}\Big(\sum_{i=1}^{r} \frac{\nu(m_i - 1)}{m_i} \Gamma_i\Big).$$

e) *The j-invariant defines a rational map $j : W \to \mathbb{P}^1$, regular in a neighborhood of $h^{-1}(S)$, and $j^*(\infty) = J_\infty$.*

By [8], lemma 1.2, 3.2 holds true if the discriminant locus is a normal crossing divisor and if one allows further blow ups. Hence one obtains 3.2 over the complement in W of finitely many points of W_0. Since X_0 is non-singular, since $g_0 : X_0 \to W_0$ is flat and since $g_{0*}\omega_{X_0/W_0}^{\nu}$ is invertible, b), c) and d) extend to W_0.

Let us recall the following property of the multiple locus Σ, first observed by Iitaka.

Lemma 3.3. *Keeping the notations introduced above, $\Sigma^{(0)} = \Sigma \cap W_0$ is étale over B_0 and the fibres of $(g_0^{-1}\Sigma^{(0)})_{\mathrm{red}} \to \Sigma^{(0)}$ are reduced.*

Proof. Let us write again $\Gamma^{(0)} = g_0^* \Sigma^{(0)}$. Then $\omega_{X_0} = g_0^*(g_{0*}\omega_{X_0}) \otimes \mathcal{O}_{X_0}(\Gamma^{(0)} - \Gamma^{(0)}_{\mathrm{red}})$.

If $\Sigma^{(0)} \to B_0$ is not étale, there exists some $b \in B_0$ such that $\Sigma^{(0)}|_{W_b}$ contains a multiple point. This remains true, if we replace B_0 by any finite cover $B_0' \to B_0$.

In particular, in order to prove the first part of 3.3, we may assume that $\Sigma^{(0)} = \sum_{i=1}^{s} \Sigma_i$, for Σ_i the image of a section of $W_0 \to B_0$. The same can be assumed for the second part. In fact, if $\Sigma^{(0)} \to B_0$ is étale but some fibre of $\Gamma^{(0)}_{\rm red} \to \Sigma^{(0)}$ non reduced, then the same remains true after replacing B_0 by an étale covering.

Consider for some $r \geq 1$ a point $v \in W_b$ which lies exactly on r of the components Σ_i of $\Sigma^{(0)}$, say

$$v \in \Sigma_1 \cap \ldots \cap \Sigma_r \cap W_b.$$

Let E denote the reduced fibre of g_b or g over v, let $\Gamma_i = (g^*\Sigma_i)_{\rm red}$ and let m_i be the multiplicity of Γ_i in $g^*\Sigma_i$. Finally let M be the multiplicity of E as a fibre of $g_b : X_b \to W_b$ and $\Gamma_i.W_b$ the intersection cycle, a positive multiple of E.

For all $\mu \geq 1$ the natural map $g_0^* g_{0*}\omega^{\mu}_{X_0} \to \omega^{\mu}_{X_0}$ induces an isomorphism

$$g_0^* g_{0*}\omega^{\mu}_{X_0} \xrightarrow{\cong} \omega^{\mu}_{X_0}\Big(-\sum_{i=1}^{s} m_i \Big\langle \frac{\mu \cdot (m_i - 1)}{m_i} \Big\rangle \Gamma_i\Big)$$

where $\langle a \rangle = a - [a]$ denotes the fractional part of a real number a. Since a similar equation holds true for g_b, one obtains

$$\sum_{i=1}^{r} m_i \Big\langle \frac{\mu \cdot (m_i - 1)}{m_i} \Big\rangle \cdot (\Gamma_i.W_b) = M \Big\langle \frac{\mu(M-1)}{M} \Big\rangle \cdot E. \tag{3.3.1}$$

Choosing for μ the lowest common multiple $l = \mathrm{lcm}(m_1, \ldots , m_r)$ the left hand side of (3.3.1) is zero, hence M divides l. Choosing $\mu = M$, one finds that each m_i divides M, hence $M = l = \mathrm{lcm}(m_1, \ldots , m_r)$. For $\mu = M - 1 = r_i \cdot m_i - 1$ one has

$$\frac{(M-1)(m_i-1)}{m_i} = r_i \cdot m_i - (r_i + 1) + \frac{1}{m_i} \quad \text{and} \quad \frac{(M-1)^2}{M} = M - 2 + \frac{1}{M}.$$

Therefore (3.3.1) implies that $\sum_{i=1}^{r} \Gamma_i.W_b = E$. This is only possible for $r = 1$ and if $\Gamma_1.W_b$ is reduced. □

Remark 3.4. Let Σ_1 be an irreducible component of the multiple locus Σ and let $\Gamma_1 = g^{-1}(\Sigma_1)_{\rm red}$. The fibres of $\Gamma_1 \cap X_0 \to \Sigma_1 \cap W_0$ are either smooth elliptic curves or Newton polygons. Assume the latter, i.e. that Σ_1 is contained in the discriminant locus. Then Γ_1 is non-normal. However, since the fibres of Γ_1 over points in $\Sigma_1 \cap W_0$ have at most ordinary double points as singularities, the non-normal locus must be étale over $\Sigma_1 \cap W_0$, hence over B_0. Altogether, replacing B_0 by an étale covering, we can assume that $\Sigma^{(0)} = \Sigma \cap W_0$ consists of sections and that the same holds true for the non-normal locus of the reduced multiple divisors.

In order to apply the vanishing stated in 2.3, we would like to restrict ourselves to semistable families $X \to B$. However, in doing so, one would have to allow W to be singular, and 3.2 would not apply. The following technical construction will serve as a replacement.

Lemma 3.5. *Let $f : X \to B$ be a family of elliptic surfaces of Kodaira dimension one, with $f_0 : X_0 \to B_0 = B - S$ smooth and relatively minimal. Assume that S consists of at least two points, if $B = \mathbb{P}^1$. Then there exists a finite covering*

$\tau : B' \to B$, with $B'_0 = \tau^{-1}(B_0)$ étale over B_0 and a diagram of projective morphisms

$$\begin{array}{ccccc}
X' & \xrightarrow{\eta'} & X^s & \xrightarrow{\sigma'} & X \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g^s} & & \downarrow{\scriptstyle g} \\
W' & \xrightarrow{\eta} & W^s & \xrightarrow{\sigma} & W \\
{\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h^s} & & \downarrow{\scriptstyle h} \\
B' & \xrightarrow{=} & B' & \xrightarrow{\tau} & B
\end{array}$$

with (as always, the index $_0$ refers to the restrictions to B_0 and B'_0):

i) *η_0 and η'_0 are isomorphisms. σ_0 and σ'_0 are fibre products.*

ii) *X', W' and X^s are non-singular, W^s is normal with at most rational Gorenstein singularities.*

iii) *$f^s = h^s \circ g^s : X^s \to B'$ is semistable, hence the fibres of $h^s : W^s \to B'$ are reduced, and for $f' = h \circ g$ the fibres $\Delta' = f'^{-1}(B' - B'_0)$ and $h'^{-1}(B' - B'_0)$ are normal crossing divisors.*

iv) *Let Σ' be the multiple locus for g' in W'. Then $\Sigma' \cap W_0$ is the disjoint union of sections, as well as the non-normal locus of $g'^*(\Sigma')_{\mathrm{red}} \cap X^0$.*

v) *$\delta' = g'_* \omega_{X'/W'}$ is invertible, and $j : W^s \to \mathbb{P}^1$ is regular in a neighborhood of $(h^s)^{-1}(B' - B'_0)$.*

vi) *Let D' denote the discriminant locus. Then $h'^{-1}(B' - B'_0) + D' + \Sigma'$ is a normal crossing divisor in a neighborhood of $h'^{-1}(B' - B'_0)$.*

vii) *$\delta'^{12} = \mathcal{O}_{W'}(\sum_{i=1}^{\ell} b_i D'_i) = \mathcal{O}_{W'}(J'_\infty + \sum_{i=s+1}^{\ell} b_i D'_i)$, where $\sum_{i=1}^{\ell} b_i D'_i$ is the discriminant divisor, defined in 3.1 (in particular, components corresponding to I_b^*, occur twice).*

viii) *Let $\Sigma'_1, \ldots, \Sigma'_r$ be the components of the multiple locus which dominate B'. Then for all $\nu > 0$ one has*

$$f'_* \omega^\nu_{X'/B'} = h'_* \Big(\omega^\nu_{W'/B'} \otimes \mathcal{O}_{W'} \Big(\sum_{i=1}^{r} \Big[\frac{\nu \cdot (m_i - 1)}{m_i} \Big] \Sigma'_i \Big) \otimes \delta'^\nu \Big).$$

ix) *Let $D'_{s+1}, \ldots, D'_{\ell'}$ be those components of $\sum_{i=s+1}^{\ell} D'_i$, which dominate B'. Then for all multiples ν of 12*

$$f'_* \omega^\nu_{X'/B'} = h'_* \Big(\omega^\nu_{W'/B'} \otimes \mathcal{O}_{W'} \Big(\sum_{i=1}^{r} \Big[\frac{\nu \cdot (m_i - 1)}{m_i} \Big] \Sigma'_i + \frac{\nu}{12} J'_\infty + \sum_{i=s+1}^{\ell'} \frac{\nu \cdot b_i}{12} D'_i \Big) \Big).$$

x) *$g^s_* \omega^\nu_{X^s/B'} = (\eta \circ g')_* \Omega^2_{X'/B'}(\log \Delta')^\nu$ and both sheaves are reflexive.*

Proof. We may assume, that $\Delta + D + \Sigma$ is a normal crossing divisor and that the j-invariant defines a morphism in a neighborhood of $h^{-1}(S)$.

We choose B' to be ramified over S of order divisible by the multiplicities of the components of $h^{-1}(S)$, and such that $X \times_B B'$ has a stable reduction $f^s : X^s \to B'$. 3.3 and 3.4 allow to assume that iv) holds true.

Choosing for W^s the normalization of $W \times_B B'$, the fibres of h^s are reduced and W^s has at most rational Gorenstein singularities. Obviously f^s factors through W^s.

W' is a desingularization of W^s, such that vi) holds true, and such that the flat relative minimal model, described in 3.2, exists over W'. If we take for X'

any desingularization of this minimal model, $g'_*\omega_{X'/B'}$ is invertible, and vii) holds true.

Up to now, we obtained the first seven properties, and we remark, that to this aim, we can replace B' by any larger covering. The last three properties will follow from the first ones.

Let D be an irreducible component of $(h^s)^{-1}(B'-B'_0)$. Since f^s is semistable, $(g^s)^{-1}(D)$ must be a reduced normal crossing divisor. In particular, the proper transform of D in W' can neither belong to the multiple locus, nor to the discriminant locus, except perhaps to the part corresponding to Newton polygons. In particular D will not be a component of $\sum_{i=s+1}^{\ell} D'_i$.

Let $(g^s_*\omega^\nu_{X^s/B'})^\vee$ be the reflexive hull. If 12 divides ν then in a neighborhood of a singular point w of W^s the sheaf $(g^s_*\omega^\nu_{X^s/B'})^\vee$ is isomorphic to

$$\omega_{W^s/B'}\otimes\mathcal{O}_{W^s}(\frac{\nu}{12}j^*(\infty)),$$

where $j : W^s \to \mathbb{P}^1$ is the j-invariant. In fact, w can not lie on transversal components of the multiple or discriminant locus, and as remarked above, all others are part of $j^*(\infty)$. From property vii) we obtain an injection

$$\eta^*((g^s_*\omega^\nu_{X^s/B'})^\vee) \xrightarrow{\subset} g'_*\omega^\nu_{X'/B'}, \tag{3.5.1}$$

hence $g^s_*\omega^\nu_{X^s/B'} = (g^s\circ\eta')_*\omega^\nu_{X'/B'} = \eta_*g'_*\omega^\nu_{X'/B'}$ is invertible for all multiples ν of 12. Moreover, since the parts of the multiple locus or of the discriminant locus, which are missing in the formula ix), are all exceptional components for η, we obtain ix), as well.

Property viii) follows from ix). For the equality in x) one just has to remark that for the fibre Δ^s of f^s over $B'-B'_0$ one has

$$\eta'_*\Omega^2_{X'/B'}(\log\Delta') = \Omega^2_{X^s/B'}(\log\Delta^s) = \omega_{X^s/B'} = \eta'_*\omega_{X'/B'}.$$

Let σ be a local section of $(g^s_*\omega^\nu_{X^s/B'})^\vee$ in a neighborhood of a point of W^s, which is blown up in W'. By (3.5.1) the 12-th power of this section is the direct image of a section of $g'_*\omega^{12\nu}_{X'/B'}$, hence of $(\omega_{W'/B'}\otimes\delta')^{12\nu}\otimes\mathcal{O}_{W'}(E)$ with $E\geq 0$ exceptional. Since δ'^{12} contains the inverse image of an invertible sheaf on W^s, σ must be the direct image of a section of $(\omega_{W'/B'}\otimes\delta')^\nu$ and we obtain the reflexivity in x) for all ν. □

Remark 3.6. Given a covering $B''\to B'$, étale over B_0, we can assume in 3.5 that B' dominates B''. In fact, in the proof of 3.5 we just used that iv) holds true, and that the ramification orders are large enough.

4. The proof of 0.1 in some special cases and the Jacobian fibration

Let $f : X\to B$ be a family of minimal elliptic surfaces of Kodaira dimension one, with $f_0 : X_0\to B_0$ smooth, and let $X \xrightarrow{g} W \xrightarrow{h} B$ be the factorization constructed in § 3.

Proof of 0.1 for smooth families of elliptic surfaces of general type over elliptic curves. If $B = B_0$ is an elliptic curve, the total space $X = X_0$ of a family of minimal elliptic surfaces is itself a minimal model, and the proof of the isotriviality is similar to the one given in [13] for families of surfaces of general type.

As in 1.4 the polarized variations of Hodge structures $R^i f_* \mathbb{C}_X$ are trivial, hence $R^i f_* \mathcal{O}_X$ is a free sheaf of degree zero, and by the Leray spectral sequence and by the Riemann Roch theorem on B and on X one obtains

$$- \frac{c_1(X).c_2(X)}{12} = \chi(\mathcal{O}_X) = \sum_{i=1}^{2} (-1)^i \chi(R^i f_* \mathcal{O}_X) =$$

$$\sum_{i=1}^{2} (-1)^i \deg(R^i f_* \mathcal{O}_X) = 0.$$

Assume that f is non-isotrivial and let $g : X \to W$ be the relative Iitaka map. 1.2 implies that $\omega_{X/B}$ is numerically effective of Kodaira-dimension 2, and by the canonical bundle formula, for ν sufficiently large and divisible, $\omega_{X/Y}^{\nu} = g^* \mathcal{A}$, with $\mathcal{A}$ ample on W, and $(g^* c_1(\mathcal{A})).c_2(X) = 0$. For a fibre W_b of h, one finds

$$(g^* c_1(\mathcal{A}(-W_b))).c_2(X) + (g^* W_b).c_2(X) = 0. \tag{4.0.1}$$

On the other hand, since X is a minimal model, [14], 3.2, implies that $c_2(X)$ is pseudo-effective. So, choosing ν large enough, none of the summands in (4.0.1) can be negative. Thus $c_2(X_b) = (g^* W_b).c_2(X) = 0$, showing that the only singular fibres of $X_b \to W_b$ are multiple fibres. One obtains

$$K_{X/B} = g^* \Big(K_{W/B} + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} \Sigma_i \Big),$$

as $\mathbb{Q}$-divisors.

By 1.6, applied to $h : W \to B$, we may assume that $W = C \times B$ and $h = pr_2$, if $g(W_b) \geq 1$. The same holds true for $W_b = \mathbb{P}^1$, since the $g_b : X_b \to W_b$ has at least three multiple fibres in that case. If $g(C) \neq 1$, for all i the images $pr_1(\Sigma_i)$ are points, contradicting the ampleness of the $\mathbb{Q}$-divisor

$$K_{W/B} + \sum_{i=1}^{r} \frac{m_i - 1}{m_i} \Sigma_i.$$

If $g(C) = 1$, then $K_W = K_{W/B} = 0$ and $0 = \deg K_{\Sigma_i} = (K_W + \Sigma_i).\Sigma_i = (\Sigma_i)^2$. Hence $(K_{W/B} + \Sigma)^2 = 0$, again contradicting the ampleness. $\square$

A relatively minimal elliptic fibration $\tilde{\gamma} : \tilde{J} \to W$ is called the Jacobian-fibration of g, if the generic fibre of $\tilde{\gamma}$ is the Jacobian of the generic fibre of g. As explained in [8], 1.4 - 1.6, even if $g : X \to W$ has a flat relative minimal model (see 3.2), one can not assume $\tilde{\gamma}$ to be flat. In fact, one has to exclude the points, where the discriminant locus has non-normal crossings, and certain types of collision points. Nevertheless, by [8], 1.6, the canonical bundle formula $\omega_{\tilde{J}} = \tilde{\gamma}^*(\omega_W \otimes \delta)$ remains true.

Assume that for $g_0 : X_0 \to W_0$ the multiple locus is empty. Since the same holds true for the fibres $X_b \to W_b = h^{-1}(b)$, each fibre of g_0 has a reduced component, and $g_0 : X_0 \to W_0$ has local sections over étale neighborhoods of all points. In this case, we may choose $\tilde{\gamma}_0 : \tilde{J}_0 \to W_0$, to be locally in the étale topology isomorphic to $g_0 : X_0 \to W_0$. In particular, $\tilde{J}_0$ is non-singular.

The same remains true, if X_0 is non-singular, but if finitely many of the fibres $X_b \to W_b$ have isolated singularities.

We choose a desingularization $\sigma : J \to \tilde{J}$ with $\sigma^{-1}(\tilde{J}_0) \cong \tilde{J}_0$. The induced family

$$\gamma = \tilde{\gamma} \circ \sigma : J \longrightarrow W$$

will be called a Jacobian fibration of g.

Lemma 4.1. *Assume that $g_0 : X_0 \to W_0$ has no multiple fibres and that*

$$X \xrightarrow{g} W \xrightarrow{h} B$$

satisfies the conditions stated in 3.5, vii) - ix) (with $B' = B$). Let $J \xrightarrow{\gamma} W$ be a Jacobian fibration. Then, using the notations from 3.5

$$\gamma_* \omega^{\nu}_{J/W} = \delta^{\nu} \quad \text{and} \quad f_* \omega^{\nu}_{X/B} = (h \circ \gamma)_* \omega^{\nu}_{J/B}$$

for all ν divisible by 12.

Proof. By the canonical bundle formula [8], 1.6, $\gamma^*\delta$ is a subsheaf of $\omega_{J/W}$. Hence δ^{ν} is an invertible subsheaf of $\gamma_*\omega^{\nu}_{J/W}$, and since both coincide outside of a finite number of points, they are the same. The second equality follows from 3.5 viii). □

Corollary 4.2. *0.1 holds true for families of elliptic surfaces of Kodaira dimension one and without multiple fibres.*

Proof. For a Jacobian fibration $\gamma : J \to W$ write $\psi = h \circ \gamma : J \to B$. By 3.5 and 3.6 we can find a covering $B' \to B$, étale over B_0, such that the conditions in 3.5 are satisfied for suitable models of both, $X \times_B B'$ and $J \times_B B'$. We will drop the $'$ and assume $B = B'$.

The family $f : X \to B$ is birational to the semistable family $f^s : X^s \to B$. If f is not birationally isotrivial, 1.2 implies that $f_*\omega^{\nu}_{X/B}$ is ample, and $\omega_{X/B}$ will be semi-ample with respect to X_0. The property viii) in 3.5 implies that, $\omega_{W/B} \otimes \delta$ is ample with respect to W_0. By 4.1, the same holds true for $\mathcal{L} = \gamma_*\omega_{J/B}$. Choose the effective divisor Υ, such that

$$\Omega^2_{J/B}(\log \psi^{-1}(S)) \cap \gamma^*\mathcal{L} = \gamma^*\mathcal{L} \otimes \mathcal{O}_J(-\Upsilon). \tag{4.2.1}$$

The last condition in 3.5 implies that, for all $\nu > 0$,

$$\psi_*\Omega^2_{J/B}(\log \psi^{-1}(S))^{\nu} = \psi_*\omega^{\nu}_{J/B},$$

hence Υ satisfies the condition vii) in 2.1. $J_0 \to B_0$ is smooth, and choosing Π as the closure of the zero-section of $J_0 \to W_0$ the assumptions in 2.1 hold true (with $T = \emptyset$). By 2.3

$$H^0(J, \Omega^2_{J/B}(\log \psi^{-1}(S)) \otimes \gamma^*\mathcal{L}^{-1} \otimes \mathcal{O}_J(\Upsilon)) = 0,$$

contradicting the choice of Υ in (4.2.1). □

Using 1.4 and some special considerations for the case that the j-invariant is constant along the fibres W_b, one can replace the reference to 2.3 in the proof of 4.2 by Saito's local Torelli theorem [17].

Corollary 4.3. *For $B_0 = \mathbb{C}^*$, 0.1 holds true if the general fibre X_b is an elliptic surface of general type, and if the Iitaka map $g_b : X_b \to W_b$ satisfies one of the following:*

a) $g(W_b) \geq 1$.

b) $W_b \cong \mathbb{P}^1$ *and* g_b *has three or more multiple fibres.*
c) $W_b \cong \mathbb{P}^1$ *and* g_b *has two multiple fibres of the same multiplicity* m.
d) $W_b \cong \mathbb{P}^1$ *and* g_b *has two smooth multiple fibres of multiplicity larger than* 6.

Sketch of the proof. We may assume that the transversal components $\Sigma_1, \ldots, \Sigma_r$ of the multiple locus are the images of sections of $X_0 \to W_0$.

In a) $W_0 \to B_0$ is an isotrivial family of curves and by 1.6 we may assume that $W_0 = C \times B_0$. Then the multiple locus is of the form $\sum_{i=1}^r c_i \times B_0$. If $W_b = \mathbb{P}^1$ we may choose an isomorphism $W_0 \cong \mathbb{P}^1 \times B_0$ with $\Sigma_i = c_i \times B_0$.

In the first three cases there exist coverings of C or $\mathbb{P}^1$ with exact ramification order m_i over c_i (see [6], IV.9.12, for example).

In case a) or c) it is easy to describe such a covering explicitly: Replacing C in a) by an étale cover of degree two, we may assume that the multiplicities of the fibres over $c_{2i} \times B_0$ and over $c_{2i+1} \times B_0$ are m_{2i}. By [5], 3.15, the covering obtained by taking the m_{2i}-th root out of the divisor $c_{2i} + (m_{2i} - 1) \cdot c_{2i+1}$ is totally ramified of order m_{2i} over $c_{2i} + c_{2i+1}$, and nowhere else. The normalization of the fibred product of the coverings obtained, is the one asked for. In c) one just takes the m-th root out of the divisor $c_1 + (m-1) \cdot c_2$.

Hence in a), b) or c) there exists a covering W_0', ramified over Σ_i of order m_i and étale over $W_0 - \bigcup_{i=1}^r \Sigma_i$. The normalization X_0' of $X_0 \times_{W_0} W_0'$ is étale over X_0, hence it remains smooth over B_0. The projection to W_0' has no multiple fibres, and 4.3 follows from 4.2 and 1.3.

For d) one shows, as indicated in 6.1, that after replacing B_0 by an étale cover, a multiple component Σ_i with multiplicity m_i gives rise to a morphism from Σ_i to the moduli scheme of elliptic curves with level m_i-structure. Since the genus of this moduli scheme is larger than one, for $m_i > 6$, this map must be constant. Hence $g^{-1}(\Sigma_i)_{\text{red}} \to \Sigma_i$ is smooth over $\Sigma_i \cap W_0$. Choose a covering $W_0' \to W_0$, ramified of order $m_1 \cdot m_2$ along $\Sigma_1 + \Sigma_2$, and nowhere else. Then the normalization of $X_0 \times_{W_0} W_0'$ is again smooth over B_0, but without multiple fibres. $\square$

Although we will reprove 4.3 in section 7, using slightly different coverings $W_0' \to W_0$, let us concentrate for a moment on those families, not covered by 4.3, a), b) or c), i.e. those with $B_0 = \mathbb{C}^*$, with $W_b = \mathbb{P}^1$ and with one of the following:

<u>Case I:</u> There are two multiple fibres of multiplicities $m_1 \neq m_2$ in $X_b \to W_b = \mathbb{P}^1$.

<u>Case II:</u> There is one multiple fibre of multiplicity m in $X_b \to W_b = \mathbb{P}^1$.

In the first case, we will replace $X \to W$ by a desingularization X' of the pullback $X \times_X W' \to W'$, where $W' \to W$ is totally ramified over $\Sigma_1 + \Sigma_2$ of order M, divisible by m_1 and m_2. Doing so, the morphism $X_0' \to B_0$ will no longer be smooth in a finite subset of X_0'. A careful analysis of the geometry of the multiple fibres in section 6 will allow to choose M in such a way, that X' locally factors through a finite morphism $X' \to X''$, with X'' smooth over B. This observation will allow to apply 2.3 to X_0', along the same lines used to prove 4.2. The sheaf $\mathcal{L}$ will correspond to the inverse image of the $\mathbb{Q}$-divisor $K_{W/B} + \sum_{i=1}^2 \frac{m_i - 1}{m_i} \Sigma_i + \delta$ on W'.

The same construction (with $m = m_1$ and $m_2 = 1$) works in case II, if one is able to choose the second section Σ_2 in such a way that it only meets components of the discriminant locus corresponding to reduced singular fibres (types I_n, II,

III or IV). To find such a section, we will have to study the discriminant locus in section 5. There we will use in an essential way that $\chi(\mathcal{O}_{X_b}) \geq 2$, a condition which fortunately holds true in case II.

5. Constantness of the Weyl system

If $\chi(\mathcal{O}_F) \geq 2$, for a general fibre F of $f : X \to B$, then the triviality of the variations of Hodge structures forces the part of the discriminant locus which corresponds to non-reduced non-multiple fibres to be étale over B_0.

Proposition 5.1. *Let $f_0 : X_0 \xrightarrow{g_0} W_0 \xrightarrow{h_0} B_0$ be a smooth projective family of minimal elliptic surfaces with $\chi(\mathcal{O}_{X_b}) \geq 2$ and $\kappa(X_b) = 1$, for all $b \in B_0$ and $X_b = f^{-1}(b)$. Assume that $B_0 = \mathbb{C}^*$ or that B_0 is an elliptic curve. Let*

$$D^{(0)} = \sum_{i=s+1}^{\ell} D_i$$

be the part of the discriminant locus in W_0, which corresponds to singular fibres of types I_j^* ($j \geq 0$), II^*, III^* or IV^*. Then $D^{(0)}$ is étale over B_0, the restriction $g_0^{-1}(D^{(0)}) \to D^{(0)}$ is locally equi-singular, and $D^{(0)} \cap \Sigma^{(0)} = \emptyset$ for the multiple locus $\Sigma^{(0)}$ of $X_0 \to W_0$.

The condition $\chi(\mathcal{O}_{X_b}) \geq 2$ is needed in the proof of the following description of the -2 classes in the Néron-Severi group $NS(X_b)$ of X_b.

Lemma 5.2. *Let $g_b : X_b \to W_b$ be a minimal elliptic surface of Kodaira dimension one with $\chi(\mathcal{O}_{X_b}) \geq 2$. Define the numbers n_i, m_j and l_k as the number of reducible fibres, according to the following list:*

type	${}_mI_i$ $(m \geq 1, i \geq 4)$	IV or ${}_mI_3$	III or ${}_mI_2$	I_j^*	II^*	III^*	IV^*
number of fibres	n_i	n_3	n_2	m_j	l_8	l_7	l_6
Euler number	i	4 or 3	3 or 2	$j+6$	10	9	8

Let $\quad N_b = \big\langle \alpha \in NS(X_b);\ (\alpha.F) = 0$ *and* $(\alpha.\alpha) = -2 \big\rangle / \mathbb{Q} \cdot F \cap NS(X_b)$

be the root lattice. Then

i) *N_b is generated by the classes of irreducible components of reducible fibres of g_b.*

ii) *The numbers n_i, m_j and l_k are uniquely determined by N_b and by its decomposition*

$$N_b \simeq \bigoplus_{i \geq 2} A_i^{\oplus n_i} \oplus \bigoplus_{j \geq 0} D_{j+4}^{\oplus m_j} \oplus \bigoplus_{k=6}^{8} E_k^{\oplus l_k}$$

in indecomposable sublattices.

Proof. The assertion ii) follows from i) and from the well-known uniqueness of the decomposition of N_b (see for example [9], Proposition 11.3).

For i) let $[D] \in NS(X_b)$ be a representative of a class $\alpha \in N$, with $(D.D) = -2$. Since K_{X_b} is numerically equivalent to $a \cdot F$, for some $a \in \mathbb{Q}$, one obtains from

the Riemann Roch formula

$$\chi(\mathcal{O}_{X_b}(D)) = \frac{D.(D - K_{X_b})}{2} + \chi(\mathcal{O}_{X_b}) = -1 + \chi(\mathcal{O}_{X_b}) > 0.$$

Therefore $H^0(\mathcal{O}_{X_b}(D)) \neq 0$ or $H^0(\mathcal{O}_{X_b}(K_{W_b} - D)) \neq 0$. Since $[K_{W_b} - D] = [-D]$ in N_b, replacing α by $-\alpha$ we may assume that α is represented by an effective divisor $\Sigma\alpha_i D_i$. Since $0 = (\alpha.F) = \Sigma\alpha_i(D_i.F)$ one finds the D_i to be irreducible components of the fibres of g_b, and one may assume that those are components of reducible fibres. One obtains i) from the classification of the singular fibres (see [1], for example). □

Proof of 5.1. Let $\mathcal{B} = \mathbb{C} \to B_0$ be the universal covering of B_0 and denote the pullback of $X_0 \to W_0 \to B_0$ by

$$\tilde{f} : \mathcal{X} \xrightarrow{\tilde{g}} \mathcal{W} \xrightarrow{\tilde{h}} \mathcal{B}.$$

Then $R^2\tilde{f}_*\mathbb{Z}_{\mathcal{X}}$ is a constant system, i.e. we have a global marking

$$\tau : R^2\tilde{f}_*\mathbb{Z}_{\mathcal{X}} \xrightarrow{\cong} H^2 \times \mathcal{B}$$

for H^2 a lattice isomorphic to $H^2(X_b, \mathbb{Z})$. Recall that any invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ defines a constant subsystem

$$c_1(\mathcal{L}|_{X_b})_{b\in\mathcal{B}} \subset R^2\tilde{f}_*\mathbb{Z}_{\mathcal{X}}.$$

In particular, if $\mathcal{H}$ is the inverse image of a relative ample invertible sheaf on $X_0 \to B_0$, we can define the constant system $(R^2\tilde{f}_*\mathbb{Z}_{\mathcal{X}})_{\mathrm{prim}} = [\mathcal{H}]^{\perp}$ in $R^2\tilde{f}_*\mathbb{Z}_{\mathcal{X}}$. Restricting τ one obtains an isomorphism

$$\tau^{\perp} : (R^2\tilde{f}_*\mathbb{Z}_{\mathcal{X}})_{\mathrm{prim}} \xrightarrow{\cong} H^{\perp} \times \mathcal{B},$$

where $H^{\perp} \subset H^2$ is a sublattice. Using $\tau^{\perp}$, we define the global period map

$$p : \mathcal{B} \longrightarrow \mathrm{Grass}(k, H^{\perp}) \quad \text{by} \quad p(b) = (\tau^{\perp}(H^0(X_b, \Omega^2_{X_b})) \subset H^{\perp} \otimes \mathbb{C}).$$

Since $\mathcal{B} = \mathbb{C}$ is a Zariski-open subset of $\mathbb{P}^1$, we may apply [19], Theorem 7.22, and we find p to be constant. Hence $\tilde{f}_*\Omega^2_{\mathcal{X}/\mathcal{B}}$ is a flat vector bundle, that is, there exists a linear subspace $T \subsetneq H^{\perp} \otimes \mathbb{C}$, such that $\tau^{\perp}$ induces an isomorphism

$$\tilde{f}_*\Omega^2_{\mathcal{X}/\mathcal{B}} \xrightarrow{\cong} T \otimes \mathcal{O}_{\mathcal{B}}.$$

We define $NS = T^{\perp} \cap H^2_{\mathbb{Z}}$ and consider the corresponding constant system

$$\tau : \mathcal{NS} \xrightarrow{\cong} NS \times \mathcal{B}.$$

By the Leftschetz (1,1) Theorem , $\mathcal{NS}$ is the system consisting fibrewise of the Néron-Severi groups $NS(X_b)$ (Note that in general, the system $NS(X_b)$ is far from being constant).

Next we consider the constant subsystem $\mathcal{C}_1$, defined by the relative dualizing sheaf $\omega_{\mathcal{X}/\mathcal{B}}$, and the sublattice $c_1 \subset H^2$, with $\tau(\mathcal{C}_1) = c_1 \times \mathcal{B}$. Of course, $\mathcal{C}_1 \subset \mathcal{NS}$ and $(c_1.c_1) = 0$. Taking the quotient we obtain

$$\tau' : \mathcal{S} = \mathcal{C}_1^{\perp}/\mathcal{C}_1 \otimes_{\mathbb{Z}} \mathbb{Q} \cap \mathcal{NS} \xrightarrow{\cong} S \times \mathcal{B} = c_1^{\perp}/c_1 \otimes_{\mathbb{Z}} \mathbb{Q} \cap NS.$$

Up to now, we obtained a constant system $\mathcal{S}$ consisting fibrewise of

$$\{a_b \in NS(X_b);\ a_b.c_1(\omega_{X_b}) = 0\}$$

modulo rational multiples of $c_1(\omega_{X_b})$. Since $\kappa(X_b)=1$, $c_1(\omega_{X_b})$ is some positive multiple of the class of a fibre $[F_b]$ of $X_b \to W_b$. Note that the intersection form on $H^2(X_b,\mathbb{Z})$ descends to the one on $\mathcal{S}$, since the condition

$$(\alpha.c_1(\omega_{X_b})) = (\beta.c_1(\omega_{X_b})) = 0$$

implies that

$$(\alpha.\beta) = ((\alpha + a\cdot c_1(\omega_{X_b})).(\beta + b\cdot c_1(\omega_{X_b}))).$$

S is a negative definite lattice. Consider the sublattice

$$N := \big\langle \alpha \in S; (\alpha.\alpha) = -2 \big\rangle$$

and the corresponding constant system

$$\mathcal{N} \xrightarrow{\cong} N \times \mathcal{B}.$$

This $\mathcal{N}$ is a family of lattices, which fibrewise corresponds to the lattice N_b described in 5.2. In particular, the numbers n_i, m_j, and l_k defined in 5.2 are independent of $b \in \mathcal{B}$, and by definition of n_i $(i \geq 4)$, m_j, and l_k one obtains

Claim 5.3. The number of singular fibres of type I_i $(i \geq 4)$, I_j^*, II^*, III^* and IV^* in $X_b \to W_b$ is independent of $b \in B_0$.

To finish the proof of 5.1 we need the constantness of the local Euler numbers. For $p \in \mathcal{W}$ choose small disks $\Delta^2_{(x,y)} \subset \mathcal{W}$ with center p, and $\Delta_x \subset \mathcal{B}$ with center $\tilde{h}(p)$, such that $\tilde{h}|_{\Delta^2_{(x,y)}} : \Delta^2_{(x,y)} \to \Delta_x$ is the projection of the first factor. For $\epsilon > 0$ sufficiently small and $\alpha \in \mathbb{C}$, with $|\alpha| < \epsilon$, we write $L_{\alpha,\epsilon} = (x - \alpha\cdot y = t)$, and $\Delta^2 = \bigcup_{|t|<\delta} L_{\alpha,t}$. Hence $\tilde{g}^{-1}(L_{\alpha,t}) \to L_{\alpha,t}$ is a family of smooth, local elliptic surfaces, parameterized by t.

The Euler numbers of $\tilde{g}^{-1}(L_{\alpha,t})$ are independent of t, and they are the sum of the Euler numbers of the singular fibres of $\tilde{g}^{-1}(L_{\alpha,t}) \to L_{\alpha,t}$.

Let again $D_1,\dots,D_\ell$ be the components of the discriminant locus and let e_i be the Euler number of the general fibre over D_i. We assume, renumbering $D_1,\dots,D_\ell$ if necessary, that $e_1 \leq e_2 \leq \dots \leq e_\ell$. Assume that for i_0 the divisor $\sum_{i=i_0+1}^{\ell} D_i$ is étale over B_0, but $\sum_{i=i_0}^{\ell} D_i$ is not. Hence there exists $p \in \mathcal{W}$, where for a suitable choice of ϵ,

$$\tilde{g}^{-1}(L_{0,0}) \longrightarrow L_{0,0}$$

has just one singular fibre, whereas the number of singular fibres in

$$\tilde{g}^{-1}(L_{0,t}) \longrightarrow L_{0,t}$$

is larger than or equal to two. Hence the Euler number of $\tilde{g}^{-1}(p)$ is strictly larger than e_{i_0}. By 5.3 this is only possible for $e_{i_0} < 4$.

Since $e_i \geq 6$, for the components corresponding to singular fibres of types I_b^*, II^*, III^*, IV^*, we obtain that the union of the corresponding components is étale over the base. Also, those components can not meet the multiple locus, since the reduced fibre of an intersection point must be a Newton polygon of length larger than or equal to 6, contradicting again 5.3. $\square$

Remark 5.4. The method used to prove 5.1 gives a bit more. The constantness of the local Euler numbers and 5.3 exclude for example, that some $p \in W_0$ lies on two components D_1 and D_2 of the discriminant locus, which correspond to fibres of type I_{b_1}, I_{b_2} with $b_1 + b_2 \geq 5$.

Nevertheless, the method is not strong enough, to imply the étaleness of the whole discriminant locus. For example it can not exclude that D_1, a component corresponding to I_1, has a cusp. Such examples exist locally, and two I_1-fibres degenerate towards a II-fibre in such a point.

6. Standard modifications of multiple fibres

Let $f_0 : X_0 \to W_0 \to B_0$ be a smooth family of minimal elliptic surfaces. We assume that the multiple locus $\Sigma^{(0)} = \sum_{i=1}^{r} \Sigma_i$ and the non-normal locus of $g_0^{-1}(\Sigma^{(0)})_{\mathrm{red}}$ consists of the union of disjoint sections. Let $m_1, \dots, m_r$ be the multiplicities of $g_0^{-1}(\Sigma_1), \dots, g_0^{-1}(\Sigma_r)$, respectively. The multiple locus can meet other components of the discriminant locus. An example, due to Moishezon, is given in [7], 7.4, where Σ_i is not contained in the discriminant locus, but meets a component D_1 of type I_n. In this example, Σ_i is an m_i-fold tangent to D_1. In fact, it is easy to show, that Σ_i can only meet the discriminant locus in components of J_∞, and this intersection can not be transversal. For components of type I_n^*, II^*, III^* and IV^*, this has been part of 5.1, at least if $B_0 = \mathbb{C}^*$ or an elliptic curve.

Consider a finite covering $W_0' \to W_0$ which is totally ramified of order m_i and étale over $W_0 - \Sigma_i$, in a neighborhood of Σ_i. The normalization X_0' of $X_0 \times_{W_0} W_0'$ is étale over X_0, and one obtains an étale Galois cover $\Gamma_i' \to \Gamma_i = g_0^{-1}(\Sigma_i)_{\mathrm{red}}$. Hence Γ_i' has a fixed point free action of $\mathbb{Z}/m_i\mathbb{Z}$, and the only singular fibres of $\Gamma_i' \to \Sigma_i' = (\Sigma_i \times_{W_0} W_0')_{\mathrm{red}}$ are smooth elliptic curves or Newton polygons of length divisible by m_i.

Remark 6.1. The j-invariant might be non constant along Σ_i. Although not needed in the sequel, let us point out some obvious obstructions for this to happen, in case $B_0 = \mathbb{C}^*$. If J_i denotes the Jacobian of Γ_i', we can assume (replacing B_0 by an étale cover) that J_i has a level m_i-structure. Hence the j-invariant factors through $\Sigma_i' \to X_1(m_i)$, where $X_1(m_i)$, is the moduli curve parameterizing elliptic curves with a level m_i-structure. By [18], 1.6.4, $g(X_1(m_i)) = 0$ implies that $m_i \leq 6$. Using the additional information, that the translation by one of the sections of order m_i can only have fixed points in two fibres (the ones over $\{0, \infty\} \subseteq \mathbb{P}^1 = B$), one can exclude the case $m_i = 6$, but not the others.

For the proof of 0.1 we will need a slightly different description of the multiple locus. For example, if $X_b \to W_b$ is an elliptic surface with two multiple fibres of multiplicities m_1 and m_2, we have to replace W_0 by a covering W_0', totally ramified of order divisible by $\mathrm{lcm}(m_1, m_2)$. The normalization of the pullback of X_0 will have no multiple fibres anymore, but it might not allow a model, smooth over B_0. So in order to apply 2.3, along the same line we did in 4.2, we have to construct a model X_0' for which we control the sheaf $\Omega^2_{X_0'/B_0}(\log \Delta')'$, defined in 2.2.

Lemma 6.2. *Assume that the multiple locus $\Sigma^{(0)}$ of*

$$X_0 \xrightarrow{g_0} W_0 \xrightarrow{h_0} B_0$$

consists of sections, as well as the non-normal locus of $g_0^{-1}(\Sigma^{(0)})_{\mathrm{red}}$. Then we can attach to each component Σ_i of $\Sigma^{(0)}$ a number μ_i, divisible by the multiplicity m_i of $\Gamma_i = g_0^{-1}(\Sigma_i)$ with the following property.

Let $\tau : W_0' \to W_0$ be a covering, totally ramified of order M, divisible by μ_i, along Σ_i and unramified over $U - \Sigma_i$ for a neighborhood U of Σ_i in W_0. Then there exists a commutative diagram of projective morphisms

$$\begin{array}{ccc} X_0' & \xrightarrow{\tau'} & X_0 \\ {\scriptstyle g_0'}\downarrow & & \downarrow{\scriptstyle g_0} \\ W_0' & \xrightarrow{\tau} & W_0 \\ {\scriptstyle h_0'}\downarrow & & \downarrow{\scriptstyle h_0} \\ B_0 & \xrightarrow{=} & B_0 \end{array}$$

such that in a neighborhood of ${g'}_0^{-1}\tau^{-1}(\Sigma_i)$ the following conditions hold true:

i) *X_0' is non-singular and τ' induces a birational morphism $X_0' \to X_0 \times_{W_0} W_0'$, biregular over $\tau^{-1}(W_0 - \Sigma^{(0)})$.*

ii) *$f_0' = h_0' \circ g_0'$ is smooth, outside of a finite number of points $t_1, \dots, t_k$.*

iii) *For each of the points t_j in ii), there exists a factorization*

$$\begin{array}{ccccc} X_0' & \xrightarrow{\sigma'} & X_0'' & \xrightarrow{\delta'} & X_0 \\ {\scriptstyle g_0'}\downarrow & & \downarrow{\scriptstyle g_0''} & & \downarrow{\scriptstyle g_0} \\ W_0' & \xrightarrow{\sigma} & W_0'' & \xrightarrow{\delta} & W_0 \\ {\scriptstyle h_0'}\downarrow & & \downarrow{\scriptstyle h_0''} & & \downarrow{\scriptstyle h_0} \\ B_0 & \xrightarrow{=} & B_0 & \xrightarrow{=} & B_0 \end{array}$$

with $h_0'' \circ g_0''$ smooth in $\sigma'(t_j)$ and σ' finite over a neighborhood of t_j.

Proof. In what follows we will work locally in U, but by abuse of notations we will write $U = W_0$ and $\Sigma = \Sigma_i$.

Consider first the case where the reduced general fibre of $\Gamma = g_0^{-1}(\Sigma) \to \Sigma$ is a smooth elliptic curve. Let us assume for a moment that $W_0'' \to W_0$ has ramification order m, the multiplicity of Γ. Then the normalization

$$\tilde{g} : \tilde{X}_0 \longrightarrow W_0'' \quad \text{of} \quad pr_2 : X_0 \times_{W_0} W_0'' \longrightarrow W_0''$$

is étale over X_0, hence smooth over B_0. However $\Gamma_{\text{red}} \simeq \tilde{\Gamma} = \tilde{g}^{-1}(\Sigma'')$, for $\Sigma'' = \tau^{-1}(\Sigma)_{\text{red}}$ might be singular. The fibres of $\tilde{\Gamma} \to \Sigma''$ are smooth elliptic curves or reduced Newton-polygons. Therefore $\tilde{\Gamma}$ has at most rational Gorenstein singularities.

Let q be one of those singularities, $p = \tilde{g}_0(q)$. We choose local parameters (x, y) on W_0 in p, such that Σ is the zero set of y and such that x is the pullback of a local parameter on B_0 in $h(p)$. So the branched cover $\tau : W_0'' \to W_0$ is locally given by $\tau^* y = w^m$ and $\tau^* x = x''$, for parameters w and x'' on W_0''.

Considering the projection given by w and the morphism induced in a neighborhood of q in $\tilde{X}_0$, we obtain a family of surfaces with a smooth general fibre and with an isolated rational Gorenstein singularity in the special fibre $w = 0$. By [3] such a singularity allows a simultaneous resolution, after taking a further branched covering, totally ramified over $w = 0$. Hence, replacing m by $m \cdot \nu$ for some $\nu = \nu(q)$ depending on q, we may assume that the normalization $\tilde{X}_0$ of $X_0 \times_{W_0} W_0''$ has a small resolution $\pi : X_0'' \to \tilde{X}_0$. In particular, ${g''}_0^{-1}(\Sigma'') = \Gamma''$

is smooth and the fibres of $\Gamma'' \to \Sigma''$ are reduced curves with at most ordinary double points as singularities, at least over a neighborhood of the given point.

To do this simultaneously for all points over Γ, we have to choose μ to be divisible by m and by $\nu(q)$ for all singular points in Γ_{red}. For any multiple M of μ let $W'_0 \to W_0$ be the corresponding covering. Locally, for the point q considered above, X'_0 can be chosen to be the covering of X''_0, totally ramified along Γ'' of order $\frac{M}{m\cdot\nu(q)}$. Since Γ'' is non-singular, X'_0 is non-singular, and i) holds true. The conditions ii) and iii) follow from the construction of X'_0.

If the general fibre of $\Gamma_{\mathrm{red}} \to \Sigma$ is a Newton polygons of length b (hence of type I_b), the construction of X'_0 is quite similar. All fibres of $\Gamma_{\mathrm{red}} \to \Sigma$ are of type I_a, for $a \geq b$. Again we start with $W''_0 \to W_0$, totally ramified of order m and with the normalization $\tilde{g} : \tilde{X}_0 \to W''_0$. The non-smooth locus of $\Gamma_{\mathrm{red}} = \tilde{\Gamma} \to \Sigma''$ consists of b disjoint sections, say $L_1, \ldots, L_b$ and of a finite number of points in $\tilde{\Gamma} - (L_1 \cup \ldots \cup L_b)$. For the latter, the argument given above works. In fact, if q is one of the isolated points, $\tilde{\Gamma} - (L_1 \cup \ldots \cup L_b)$ has again a rational Gorenstein singularity in q, and choosing a larger covering there exists a simultaneous resolution.

Along L_i, the morphism $\tilde{\Gamma} \to \Sigma''$ is equi-singular. Hence for $W'_0 \to W''_0$ totally ramified over Σ'', we can simultaneously resolve the singularities in the normalization of $X''_0 \times_{W'_0} W''_0$ which are lying over L_i. □

The morphism f'_0 constructed in 6.2 is smooth outside of a finite number of points $t_1, \ldots, t_k$. As in 2.2, for $T = \{t_1, \ldots, t_k\}$, let $(\Omega^2_{X'_0/B_0})' = \omega'_{X'_0/B_0}$ be the image of $\Omega^2_{X'_0} \to \omega_{X'_0/B_0}$.

Corollary 6.3. *For M and τ' as in 6.2 and for $U' = (g_0 \circ \tau')^{-1}(U)$, one has a natural inclusion*

$$\varphi : {\tau'}^{*}\omega_{X_0/B_0}|_{U'} \longrightarrow \omega'_{X'_0/B_0}|_{U'}.$$

Proof. Both sheaves are subsheaves of $\omega'_{X'_0/B_0}$, hence in order to show that the inclusion ${\tau'}^{*}\omega_{X_0/B_0} \to \omega_{X'_0/B_0}$ factors through $\omega'_{X'_0/B_0}$, we can argue locally in a neighborhood of $t_j \in T$.

Using the notation from 6.2 iii), one has (locally in a neighborhood of t_j) a diagram

$$\begin{array}{ccccc}
{\sigma'}^{*}\Omega^2_{X''_0} & \twoheadrightarrow & {\sigma'}^{*}\omega_{X''_0/B_0} & \xleftarrow{\subset} & {\delta'}^{*}{\sigma'}^{*}\omega_{X_0/B_0} \\
\downarrow{\subset} & & \downarrow{\subset} & & \\
\Omega^2_{X'_0} & \twoheadrightarrow & \omega'_{X'_0/B_0} & \xrightarrow{\subset} & \omega_{X'_0/B_0}
\end{array}$$

and 6.3 holds true. □

Remark 6.4. The corollary 6.3 obviously remains true in the following situation: Let $\Theta \subset W_0$ be a section, such that $g_0^{-1}(\Theta)$ is non-singular and such that $g_0^{-1}(\Theta) \to \Theta$ is smooth outside of a finite number of points. Let $W'_0 \to W_0$ be (locally near Θ) a covering, totally ramified of order M along Θ and unramified elsewhere. Then $X'_0 = X_0 \times_{W_0} W'_0$ is non singular, $X'_0 \to B_0$ is smooth outside of a finite number of points, and there is (locally) a natural inclusion

$$\varphi : pr_1^{*}\omega_{X_0/B_0} \longrightarrow \omega'_{X'_0/B_0}.$$

7. The proof of 0.1 for families of elliptic surfaces

Let $f : X \to B$ be a family of minimal elliptic surfaces, smooth over $B_0 = B - S$ and with $\kappa(X_b) = 1$ for $b \in B_0$. At the beginning of section 4 we proved that f is birationally isotrivial, in case $B = B_0$ is an elliptic curve. Hence we will restrict ourselves to the case $B_0 = \mathbb{C}^*$, in this section.

By 4.3 we only have to consider families of elliptic surfaces with one or two multiple fibres. Nevertheless, since the arguments used here apply to all other cases as well, we will not make this restriction.

For the relative Iitaka fibration $X_0 \xrightarrow{g_0} W_0 \xrightarrow{h_0} B_0$, constructed in section 3, $h_0 : W_0 \to B_0$ is a smooth family of curves. By 1.6 we may write $W_0 = C \times B_0$, and 3.3 allows to assume that the multiple locus $\Sigma^{(0)}$ in W_0 consists of disjoint sections. Replacing C by an étale cover, and using 1.3 we are allowed to assume that $g_b : X_b \to W_b = C$ has more than 2 multiple fibres, provided $g(W_b) \geq 1$.

The further construction depends on the number and type of the singular fibres:

Case I: The degree r of $\Sigma^{(0)}$ over B_0 is larger than or equal to 2.

Case II: If $r = 1$, we have to find a second section Θ, with $g_0^{-1}(\Theta) \to \Theta$ smooth outside of a finite number of points, and with $g_0^{-1}(\Theta)$ non-singular.

For $g(W_b) > 0$, we were allowed to assume that $r > 1$, hence it is sufficient to construct Θ for $W_0 \simeq \mathbb{P}^1 \times B_0$. Then the canonical bundle formula and the assumption $\kappa(X_b) = 1$ imply that $\deg(g_{b*}\omega_{X_b/W_b}) = \chi(\mathcal{O}_{X_b}) \geq 2$. Depending on the singular fibres of $X_b \to W_b$, for $b \in B_0$ in general position, we distinguish two subcases:

Case II a: All non-multiple singular fibres of $X_b \to W_b$ are reduced. Hence the only singular fibres are of type ${}_mI_n$, II, III or IV. We choose for Θ a general section of $W_0 \to B_0$, not meeting Σ_1.

Case II b: If $X_b \to W_b$ has singular fibre of type I_n^*, II^*, III^* or IV^*, we have to be more careful. By 5.1 the I_n^*, II^*, III^* and IV^*-loci are disjoint in W_0, and étale over B_0. Replacing B_0 again by an étale cover we find sections $D_{s+1}, \ldots, D_{\ell'}$, corresponding to singular fibres of type I_n^*, II^*, III^* or IV^*. The isomorphism $W_0 \simeq \mathbb{P}^1 \times B_0$ can be chosen, such that $pr_1(\Sigma_1)$, and $pr_1(D_i)$ are points in $\mathbb{P}^1$, necessarily distinct.

In fact this is obvious, for $\ell' = s+1$ or $\ell' = s+2$. If $\ell' > s+2$, we choose the isomorphism such that $pr_1(\Sigma_1)$, $pr_1(D_{s+1})$ and $pr_1(D_{s+2})$ are points. Since $D_i \simeq \mathbb{C}^*$, for $i = s+1, \ldots, \ell'$, and since the D_i can not meet each other or Σ_1, the restriction $pr_1|_{D_i}$ can not be dominant, for $i > s+2$.

For Θ we choose the fibre of pr_1 over a point in general position in $\mathbb{P}^1$.

Let μ_i be the number, attached to the component Σ_i in 6.2 and let $M = \mathrm{lcm}\{\mu_1, \ldots, \mu_r\}$. We choose a covering $\tau_0 : W_0' \to W_0$, totally ramified of order M over $\Sigma_1 + \Theta$, in case II, or ramified in each component of $\displaystyle\sum_{i=1}^{r} \tau_0^*\Sigma_i$ of order M, in case I, and we assume τ_0 to be unramified elsewhere.

Such coverings exist by [6], IV.9.12, for example. As explained in [5], 3.5 and 3.15, they can also be obtained by taking the m-root out of divisors A, with $\mathcal{O}_{W_0}(A)$ the m-th power of an invertible sheaf.

In case II, choose $A = \Sigma_1 + (M-1)\cdot\Theta$, and in case I, if r is even,

$$A = \sum_{i=1}^{r/2} \Sigma_{2i-1} + (M-1)\cdot\Sigma_{2i}$$

will give the covering needed. If r and M are both odd, one can take the M-th root out of

$$A = \Sigma_{r-2} + \Sigma_{r-1} + (M-2)\cdot\Sigma_r + \sum_{i=1}^{(r-3)/2} \Sigma_{2i-1} + (M-1)\cdot\Sigma_{2i}.$$

If M is even, and r odd, take first the M-th root out of $A = \Sigma_1 + (M-1)\cdot\Sigma_2$. On the covering obtained there are $M\cdot(r-2)$ points left, and we proceed as in the first step.

Let $X'_0 \to W'_0$ be the model from 6.2 over a neighborhood of $\tau_0^{-1}(\Sigma^{(0)})$, in both cases, and equal to $X_0 \times_{W_0} W'_0$ over Θ, in case II. We constructed a non-singular variety X'_0 and projective morphisms

$$\begin{array}{ccc} X'_0 & \xrightarrow{\tau'_0} & X_0 \\ {\scriptstyle g'_0}\big\downarrow & & \big\downarrow{\scriptstyle g_0} \\ W'_0 & \xrightarrow{\tau_0} & W_0 \end{array}$$

such that $f'_0 = h_0 \circ \tau_0 \circ g'_0$ is smooth outside of a finite number of points, and such that locally in those points 6.3 (see also 6.4) holds. This remains true if we replace B_0 by further étale coverings. Hence we may choose non-singular projective compactifications

$$\begin{array}{ccc} X' & \xrightarrow{\tau'} & X \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ W' & \xrightarrow{\tau} & W \\ {\scriptstyle h'}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ B & \xrightarrow{=} & B \end{array}$$

such that $f' = h' \circ g'$ as well as $f = h \circ g$ satisfy the conditions stated in 3.5 (for $B' = B$).

Assume that $f : X \to B$ is not birationally isotrivial. Then 1.2 implies that $\kappa(\omega_{X/B}) = 2$. Using the notations from 3.5, viii), (with all the $'$ omitted), one has

$$f_*\omega^\nu_{X/B} = h_*\Big(\omega^\nu_{W/B} \otimes \mathcal{O}_W\Big(\sum_{i=1}^{r}\Big[\frac{\nu\cdot(m_i-1)}{m_i}\Big]\cdot\Sigma_i\Big)\otimes\delta^\nu\Big).$$

Hence, if ν is a multiple of $\mathrm{lcm}\{m_1,\ldots,m_r\}$, the sheaf

$$\mathcal{L}^{(\nu)} = \omega^\nu_{W/B} \otimes \mathcal{O}_W\Big(\sum_{i=1}^{r}\frac{\nu\cdot(m_i-1)}{m_i}\Sigma_i\Big)\otimes\delta^\nu$$

will be ample with respect to W_0, and, for some effective divisor R on X supported in $f^{-1}(S)$, one has

$$g^*\mathcal{L}^{(\nu)} = \omega_{X/B}(-R)^\nu.$$

Since τ is totally ramified over Σ_i of order divisible by m_i, there is an invertible sheaf $\mathcal{L}'$ on W', ample with respect to W'_0, with $\tau^*\mathcal{L}^{(\nu)} = \mathcal{L}'^{\nu}$. Moreover $g'^*\mathcal{L}' = \tau'^*\omega_{X/B}(-R)$, and $\mathcal{L}'$ is a subsheaf of $g'_*\omega_{X'/B}$.

Let $\gamma' : J' \to W'$ be a Jacobian fibration, as considered in section 4. By 4.1 one has

$$\gamma'_*\omega^{\nu}_{J'/W'} = g'_*\omega^{\nu}_{X'/W'},$$

for all $\nu \geq 1$. Replacing a last time B_0 by an étale cover, 3.6 allows to assume that $J' \to B$ satisfies the conditions stated in 3.5 (over $B = B'$).

Since $\gamma'_0 : J'_0 \to W'_0$ is locally in the étale topology isomorphic to $g'_0 : X'_0 \to W'_0$, the morphism $J'_0 \to B_0$ is again smooth outside of a finite subset T.

$J'_0 \to W'_0$ has a zero-secton with image Π_0. Writing $\psi' = h' \circ \gamma'$ and $\psi'^{-1}(S) = \nabla'$, we may assume that the closure Π of Π_0 is non-singular and that $\nabla' + \Pi$ is a normal crossing divisor.

As in 2.2 one defines

$$\Omega^2_{X'/B}(\log \Delta')' = \mathrm{Im}(\Omega^2_{X'}(\log \Delta') \longrightarrow \Omega^2_{X'/B}(\log \Delta')^{\sim}), \qquad \text{and}$$
$$\Omega^2_{J'/B}(\log \nabla')' = \mathrm{Im}(\Omega^2_{J'}(\log \nabla') \longrightarrow \Omega^2_{J'/B}(\log \nabla')^{\sim}).$$

$\omega'_{X'/B}$ denotes the subsheaf of $\omega_{X'/B}$, generated by $\Omega^2_{X'/B}(\log \Delta')'$ and by $\omega_{X'/B}$, restricted to a neighborhood of Δ'. Correspondingly $\omega'_{J'/B}$ is generated by the sheaves $\Omega^2_{J'/B}(\log \nabla')'$ and $\omega_{J'/B}|_{J'-T}$.

Since $\gamma'_0 : J'_0 \to W'_0$ and $g'_0 : X'_0 \to W'_0$ are locally isomorphic in the étale topology, the natural isomorphism

$$\gamma'_{0*}\omega_{J'/B} \xrightarrow{\simeq} g'_{0*}\omega_{X'/B}$$

induces an isomorphism

$$\gamma'_{0*}\omega'_{J'/B} \xrightarrow{\simeq} g'_{0*}\omega'_{X'/B}. \tag{7.0.1}$$

By 6.3 (see also 6.4), $g'^*\mathcal{L}' = \tau'^*\omega_{X/B}(-R)$ is contained in $\omega'_{X'/B}$. Hence $\mathcal{L}'$ lies in $g'_*\omega'_{X'/B}$, and, using the isomorphism (7.0.1) one finds an injection

$$\gamma'^*\mathcal{L}' \xrightarrow{\subset} \omega'_{J'/B}.$$

The second sheaf contains $\Omega^2_{J'/B}(\log \nabla')'$ and for some effective divisor Υ on J', supported in ∇',

$$\Omega^2_{J'/B}(\log \nabla')' \cap \gamma'^*\mathcal{L}' = \gamma'^*\mathcal{L} \otimes \mathcal{O}_{J'}(-\Upsilon). \tag{7.0.2}$$

By the last condition in 3.5, for all $\nu > 0$,

$$\psi'_*\Omega^2_{J'/B}(\log \nabla')^{\nu} = \psi'_*\omega^{\nu}_{J'/B},$$

hence

$$\psi'_*(\gamma'^*\mathcal{L}'^{\nu} \otimes \mathcal{O}_{J'}(-\nu \cdot \Upsilon)) = h'_*\mathcal{L}'^{\nu}.$$

Altogether $\mathcal{L}'$, Π and Υ satisfy the assumptions made in 2.1, for $J' \to W' \to B$. By 2.3

$$H^0(J', \Omega^2_{J'/B}(\log \nabla')' \otimes \gamma'^*\mathcal{L}^{-1} \otimes \mathcal{O}_{J'}(\Upsilon)) = 0,$$

contradicting the choice of Υ in (7.0.2). $\square$

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